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Triangular arrays of digits

by

A. van Wijngaarden

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Preface

This note concerns a problem which was suggested to the author while listening to a lecture by J.C.P. Miller at the IBM Symposium on Utilisation of Computers in Mathematical Research held at Blaricum, August 29 to 31, 1966
Dr Miller's problem was actually a different one, but the present one is intimately related with it and the result may shed some light on similar problems

Triangular arrays of digits

Let T_n be the set of triangular arrays D of binary digits $d_{i,j}$, i.e. $d_{i,j} = 0$ or $d_{i,j} = 1$; $i = 1, \ldots, n + 1 - j$; $j = 1, \ldots, n$, that satisfy the relations

$$d_{i,j} + d_{i+1,j} + d_{i,j+1} \equiv 0 \pmod 2 \quad (1)$$

for all relevant values of i and j.

An array D may be pictured as an equilateral triangular array of 0's and 1's, the i-coordinate counting from left to right and the j-coordinate upwards (cf. fig. 1).

A reflection M with respect to the altitude through the apex and also a counter-clockwise rotation over 120^o, R, do not only transform the array into a similarly constructed one, but also leave the relation (1) intact. Indeed, (1) states that on each elementary triangle with its apex pointing upwards, there occur either one or three zero digits, a property which is invariant under reflection and rotation. Hence, with D also MD, RD, MRD, RRD and MRRD are, not necessarily different, elements of D. Our problem is to determine the number t(n) of essentially different elements of T_n, where by essentially different is meant not transformable into one another by a number of rotations and or or reflections.

An array D is completely defined by its base, i.e. the sequence $d_{1,1}, d_{2,1}, \ldots, d_{n,1}$, since by means of (1) the remaining digits can be determined successively. Hence, the number of elements of T_n is the number of different bases, i.e. 2^n.

The number of essentially different elements in the subset (D, MD, RD, MRD, RRD, MRRD) is called the rank of that subset, and may be either 1, 2, 3 or 6. If the elements possess a centre of symmetry and an axis of symmetry as well then the rank is 1; if they only possess a centre of symmetry then the rank is 2; if they only possess one axis of symmetry then the rank is 3; otherwise the rank is 6. In fig. 2, 3, 4 and 5 a number of arrays of order 4 and of different rank are pictured. Since the total number of bases in these figures is 16, actually, all arrays of order 4 are depicted. Let $t_1(n)$, $t_2(n)$, $t_3(n)$ and $t_6(n)$ denote the number of essentially different arrays of order n and of rank 1, 2, 3 and 6 respectively. Thus, e.g. $t_1(4) = 2$, $t_2(4) = 1$, $t_3(4) = 2$, $t_6(4) = 1$, and $t(4) = 6$. Obviously, in general,

$$t_1(n) + t_2(n) + t_3(n) + t_6(n) = t(n). \tag{2}$$

Moreover, since an array of rank r disposes over r different bases,

$$t_1(n) + 2t_2(n) + 3t_3(4) + 6t_6(n) = 2^n. \tag{3}$$

First the number of arrays, of order n and of rank either 1 or 3 is determined. An array of order 2n + 1 and of rank 1 or 3 consists of an array of order 2n and of rank 1 or 3 "on top" of a symmetric sequence of 2n + 1 digits, of course, satisfying (1). Since the middle digit of this sequence can be chosen freely to be either 0 or 1 (cf. Fig. 6) one has

$$t_1(2n + 1) + t_3(2n + 1) = 2(t_1(2n) + t_3(2n)). \tag{4a}$$

Half of the arrays thus produced have as middle digit a 0, half of them a 1. By "underlining" an array of order 2n and of rank 1 or 3 having a 0 as middle digit two essentially different arrays of order 2n + 1 and of rank 1 or 3 are obtained (cf. fig. 6). By underlining an array of order 2n and of rank 1 or 3 having a 1 as middle digit one does obtain two arrays of order 2n + 1 and of rank 6 which are identical but for a reflection. Hence,

$$t_1(2n + 2) + t_3(2n + 2) = t_1(2n + 1) + t_3(2n + 1). \tag{4b}$$

Since, moreover, $t_1(0) = 1$ and $t_3(0) = 0$ one has

$$t_1(2n) + t_3(2n) = 2^n,$$
$$t_1(2n + 1) + t_3(2n + 1) = 2^{n + 1}. \tag{4}$$

An array of order 2n + 3 and of rank 1, is obtained by "lining" an array of order 2n and of rank 1, i.e. by extending it on all three sides with the same sequence of digits, of course, satisfying (1) (cf. fig. 7). On the same grounds as above one obtains two essentially different arrays, one having a 0 as middle digit of its sides and one having a 1 as middle digit. Hence,

$$t_1(2n + 3) = 2t_1(2n). \tag{5a}$$

Lining an array of order 2n + 1 and of rank 1 and having a 0 as middle digit of its sides yields two essentially different arrays of order 2n + 4 and of rank 1 (cf. fig. 7). Lining an array of order 2n + 1 and of rank 1 and having a 0 as middle digit yields two arrays of order 2n + 4 and of rank 2, which are, moreover, identical but for a reflection. Hence

$$t_1(2n + 4) = t_1(2n + 1), \tag{5b}$$

whereas half of this amount is a contribution to $t_2(2n + 4)$.
Since, moreover, $T_1(0) = 1$, $T_1(1) = 2$ and $T_1(2) = 1$ one has

$$\begin{aligned} t_1(6k) &= t_1(6k + 2) = 2^k, \\ t_1(6k + 1) &= t_1(6k + 3) = t_1(6k + 4) = t_1(6k + 5) = 2^{k+1}, \end{aligned} \tag{5}$$

and in view of (4)

$$\begin{aligned} t_3(6k) &= 2^{3k} - 2^k, \\ t_3(6k + 1) &= 2^{3k+1} - 2^{k+1}, \\ t_3(6k + 2) &= 2^{3k+1} - 2^k, \\ t_3(6k + 3) &= t_3(6k + 4) = 2^{3k+2} - 2^{k+1}, \\ t_3(6k + 5) &= 2^{3k+3} - 2^{k+1}. \end{aligned} \tag{6}$$

The other way to produce an array of rank 2 is by lining another array of rank 2 (cf. fig. 8). This yields two arrays which are identical but for a reflection. Hence,

$$t_2(2n + 3) = 2t_2(2n), \tag{7a}$$

$$t_2(2n + 4) = 2t_2(2n + 1) + \tfrac{1}{2}t_1(2n + 4) \tag{7b}$$

Since, moreover, $t_2(0) = t_2(1) = t_2(2) = 0$, one has in view of (5):

$$\begin{aligned} t_2(6k) &= t_2(6k + 2) = 2^{2k-1} - 2^{k-1}, \\ t_2(6k + 1) &= t_2(6k + 3) = t_2(6k + 5) = 2^{2k} - 2^k, \\ t_2(6k + 4) &= 2^{2k+1} - 2^k. \end{aligned} \tag{7}$$

From (3), (5), (6) and (7) follows $t_6(n)$ and then from (2):

$$\begin{aligned} t(6k) &= (2^{6k-1} + 3 \times 2^{3k-1} + 2^{2k}) \div 3, \\ t(6k + 1) &= (2^{6k} + 3 \times 2^{3k} + 2^{2k+1}) \div 3, \\ t(6k + 2) &= (2^{6k+1} + 3 \times 2^{3k} + 2^{2k}) \div 3, \\ t(6k + 3) &= (2^{6k+2} + 3 \times 2^{3k+1} + 2^{2k+1}) \div 3, \\ t(6k + 4) &= (2^{6k+3} + 3 \times 2^{3k+1} + 2^{2k+2}) \div 3, \\ t(6k + 5) &= (2^{6k+4} + 3 \times 2^{3k+2} + 2^{2k+1}) \div 3. \end{aligned} \tag{8}$$

It is seen that $t(n) \sim 2^n \div 6$. Hence, with increasing n the fraction of arrays of rank 1, 2 or 3 tends to zero, as might be expected.

In Table I values of $t_1(n)$, $t_2(n)$, $t_3(n)$, $t_6(n)$ and $t(n)$ are displayed for small values of n.

The problem is easily generalized. One may, for instance, let T_n be the set of arrays D of digits $d_{i_1, i_2, \ldots, i_k}$ in the scale of a, i.e.
$0 \leq d_{i_1, i_2, \ldots, i_k} < a$, $i_1 = 1, \ldots, n + k - 1 - i_2 - \ldots - i_k$;
$i_2 = 1, \ldots, n + k - 2 - i_3 - \ldots - i_k$; ...; $i_k = 1, \ldots, n$,
that satisfy the relations

$$d_{i_1, i_2, \ldots, i_k} + d_{i_1 + 1, i_2, \ldots, i_k} + d_{i_1, i_2 + 1, \ldots, i_k} + \ldots + d_{i_1, i_2, \ldots, i_k + 1} \equiv r (\mathrm{mod}\ a),$$

$0 \leq r < a$, for all relevant values of $i_1, i_2, \ldots, i_k$.

In principle, the same methods may be used, but the complexity soon gets tremendous with increasing dimension and scale. Still very interesting is the case a = 2, as before, but in several dimensions. The residu r is either 0 or 1. If the number of dimensions is even then the set defined by r = 0 is intrinsically the same as that defined by r = 1, i.e. each array of the one set is transformed into one of the other set by changing every 0 into 1 and every 1 into 0. If the number of dimensions is odd, however, the two sets are intrinsically different, i.e. each array in a set is transformed into one of the same set by changing every 0 into a 1 and every 1 into a 0. The one dimensional case, k = 1, is, of course, trivial. The case k = 3 is already quite complex. The array is now tetrahedronal and the transformation group is of order 12. There are, therefore, arrays of rank 1, 2, 3, 4, 6 and 12. For some results, cf. Tables II and III.

$$
\begin{array}{ccccccc}
 & & & d_{1,4} & & & \\
 & & d_{1,3} & & d_{2,3} & & \\
 & d_{1,2} & & d_{2,2} & & d_{3,2} & \\
d_{1,1} & & d_{2,1} & & d_{3,1} & & d_{4,1}
\end{array}
$$

fig. 1 Array of order 4.

0	0	1	1	0	0
0 0	0 0	1 0	0 1	1 1	1 1
1 1 1	1 1 1	0 1 1	1 1 0	0 1 0	0 1 0
0 1 0 1	1 0 1 0	0 0 1 0	0 1 0 0	1 1 0 0	0 0 1 1
D1	MD1	RD1	MRD1	R^2D1	MR^2D1

fig. 2 Arrays of order 4 and of rank 6.

0	1	1	0	1	1
1 1	1 0	0 1	0 0	0 1	1 0
1 0 1	1 0 0	0 0 1	0 0 0	0 0 1	1 0 0
1 0 0 1	0 1 1 1	1 1 1 0	1 1 1 1	0 0 0 1	1 0 0 0
D2	RD2	R^2D2	D3	RD3	R^2D3

fig. 3 Arrays of order 4 and of rank 3.

1	1
1 0	0 1
0 1 1	1 1 0
1 1 0 1	1 0 1 1
D4	MD4

fig. 4 Arrays of order 4 and of rank 2.

0	0
0 0	1 1
0 0 0	1 0 1
0 0 0 0	0 1 1 0
D5	D6

fig. 5 Arrays of order 4 and of rank 1.

0	0	0	0	0	0
1 1	1 1	0 0	0 0	1 1	1 1
1 0 1	1 0 1	0 0 0	0 0 0	1 0 1	1 0 1
1 0 0 1	1 0 0 1	0 0 0 0	0 0 0 0	0 1 1 0	0 1 1 0
1 0 0 0 1	0 1 1 1 0	0 0 0 0 0	1 1 1 1 1	1 1 1 1 1	0 0 1 0 0
D2	D2	D5	D5	D6	D6

fig. 6 Arrays of order 5 and of rank 1 or 3 formed by underlining those of order 4 and of rank 1 or 3.

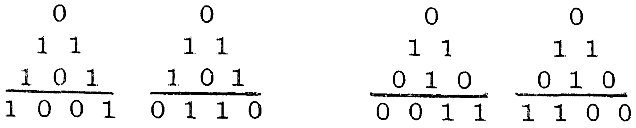

fig. 7 Arrays of order 4 and of rank 1, 3 or 6 formed by underlining arrays of order 3 and of rank 1 or 3.

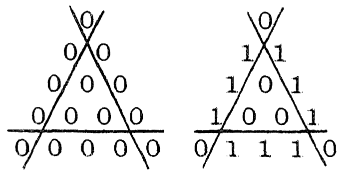

fig. 8 Arrays of order 5 and of rank 1 formed by lining arrays of order 2 and of rank 1

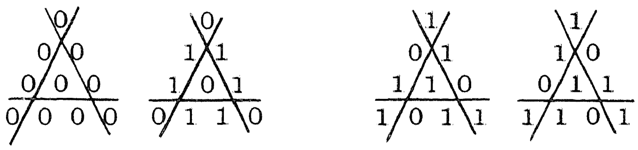

fig. 9 Arrays of order 4 and of rank 1 or 2 formed by lining arrays of order 1 and of rank 1.

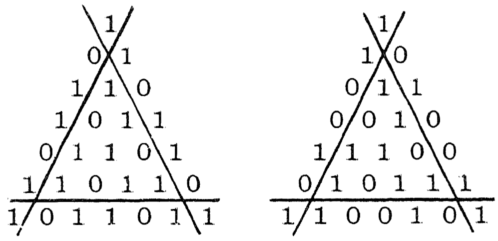

fig. 10 Arrays of order 7 and of rank 2 formed by lining arrays of order 4 and of rank 2.

Table I, a = 2, R = 0 or 1, k = 2.

n	$t_1(n)$	$t_2(n)$	$t_3(n)$	$t_6(n)$	t(n)
0	1	0	0	0	1
1	2	0	0	0	2
2	1	0	1	0	2
3	2	0	2	0	4
4	2	1	2	1	6
5	2	0	6	2	10
6	2	1	6	7	16
7	4	2	12	14	32
8	2	1	14	35	52
9	4	2	28	70	104
10	4	6	28	154	192
11	4	2	60	310	376
12	4	6	60	650	720
13	8	12	120	1300	1440
14	4	6	124	2666	2800
15	8	12	248	5332	5600
16	8	28	248	10788	11072
17	8	12	504	21588	22112
18	8	28	504	43428	43968
19	16	56	1008	86856	87936
20	8	28	1016	174244	175296
21	16	56	2032	348488	350592
22	16	120	2032	697992	700160
23	16	56	4080	1396040	1400192

Table II, a = 2, r = 0, k = 3.

n	$t_1(n)$	$t_2(n)$	$t_3(n)$	$t_4(n)$	$t_6(n)$	$t_{12}(n)$	t(n)
0	1	0	0	0	0	0	1
1	2	0	0	0	0	0	2
2	2	0	0	0	1	0	3
3	2	0	2	2	4	2	12

Table III, $a = 2$, $r = 1$, $k = 3$.

n	$t_1(n)$	$t_2(n)$	$t_3(n)$	$t_4(n)$	$t_6(n)$	$t_{12}(n)$	$t(n)$
0	1	0	0	0	0	0	1
1	2	0	0	0	0	0	2
2	0	0	0	2	0	0	2
3	2	0	2	2	0	4	10